SOLUTIONS TO SELECTED QUESTIONS IN HOMEWORK 16

MATH 241

13.3.5

Proof. Assume $u(x, t) = X(x)T(t)$, then the equation becomes $kX''T - hXT = XT'$, $kX''T = X(hT + T')$,

$$\frac{X''}{X} = \frac{hT + T'}{kT}$$

This constant depends on neither x nor t, so we can assume it is equal to $-\lambda$ a constant. Then $X'' + \lambda X = 0$, $T' + hT = -\lambda kT$, $T' = -(h + \lambda k)T$. The boundary conditions are $\frac{\partial u}{\partial x}(0, t) = 0$, $\frac{\partial u}{\partial x}(L, t) = 0$ because it says the ends are insulated. This means $X'(0) = X'(L) = 0$. Therefore the Sturm-Liouville problem is

$$X'' + \lambda X = 0, X'(0) = X'(L) = 0$$

(1) If $\lambda > 0$, the general solution to the ODE is $X(x) = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x$, $X'(x) = -C_1\sqrt{\lambda}\sin\sqrt{\lambda}x + C_2\sqrt{\lambda}\cos\sqrt{\lambda}x$. Then look at the boundary conditions, $X'(0) = 0$ implies $C_2 = 0$, and $X'(L) = 0$ means $C_1\sqrt{\lambda}\sin\sqrt{\lambda}L = 0$, therefore $\sqrt{\lambda}L = n\pi$ are the eigenvalues, and the eigenfunctions are $\lambda = \frac{n^2\pi^2}{L^2}$, $n = 1, 2, \cdots$.

(2) If $\lambda = 0$, the general solution to the ODE is $X(x) = C_1 + C_2x$, $X'(x) = C_2$, then look at the boundary conditions, $X'(0) = X'(L) = C_2$, so $C_2 = 0$, $\lambda = 0$ is an eigenvalue, and the eigenfunction is $X(x) = C$.

(3) If $\lambda < 0$, the general solution to the ODE is $X(x) = C_1 \cosh\sqrt{-\lambda}x + C_2 \sinh\sqrt{-\lambda}x$, $X'(x) = C_1\sqrt{-\lambda}\cosh\sqrt{-\lambda}x + C_2\sqrt{-\lambda}\sinh\sqrt{-\lambda}x$, then look at the boundar conditions $X'(0) = C_1 = 0$, and $X'(L) = C_2\sqrt{-\lambda}\sinh\sqrt{-\lambda}L = 0$, but $\sinh\sqrt{-\lambda}L = 0$ if and only if $\sqrt{-\lambda}L = 0$, but that is impossible. $\sqrt{-\lambda} \neq 0$, too, so $C_2 = 0$. Thus we only have trivial solution, too. So there is no negative eigenvalues.

Therefore the eigenvalues are $\lambda = \frac{n^2\pi^2}{L^2}$, $n = 0, 1, 2, \cdots$, eigenfunctions are $\cos\frac{n\pi}{L}x$, note that since $\cos 0 = 1$ so we can combine the eigenfunctions in (1) and (2). Correspondingly, $T' = -(h + \lambda k)T$ implies $\frac{T'}{T} = -(h + \lambda k)$, therefore $\ln T = -(h + \lambda k)t + C$, $T = Ce^{-(h+\lambda k)t}$. Plug in $\lambda = \frac{n^2\pi^2}{L^2}$, one gets $T = Ce^{-(h+\frac{n^2\pi^2}{L^2}k)t}$ Therefore the solution has the form

$$C_0 + \sum_{n=1}^{\infty} C_n e^{-(h+\frac{n^2\pi^2}{L^2}k)t} \cos\frac{n\pi}{L}x$$

Restrict to $t = 0$, by the initial condition one gets $f(x) = C_0 + \sum_{i=1}^{n} C_n \cos \frac{n\pi}{L} x$, so by the formula of Fourier cosine series, $C_0 = \frac{1}{2}\frac{2}{L}\int_0^L f(x)dx$, $C_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi}{L}xdx$, thus the solution is:

$$u(x,t) = \frac{1}{L}\int_0^L f(x)dx + \sum_{n=1}^{\infty}(\frac{2}{L}\int_0^L f(x)\cos\frac{n\pi}{L}xdx)e^{-(h+\frac{n^2\pi^2}{L^2}k)t}\cos\frac{n\pi}{L}x$$

□

13.2.8

Proof. The ends are secured means $u(0,t) = u(L,t) = 0$, initially the string is undisplaced means $u(x,0) = 0$, with an initial velocity $\sin\frac{\pi x}{L}$ means $\frac{\partial u}{\partial t}(x,0) = \sin\frac{\pi x}{L}$. So the BVP is:

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}$$

$$u(0,t) = u(L,t) = 0 \;\; (BC)$$

$$u(x,0) = 0, \frac{\partial u}{\partial t}(x,0) = \sin\frac{\pi x}{L} \;\; (IC)$$

□

15.4.3

Proof. It is a Dirichlet boundary condition, so use Fourier sine transform to $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$, we get

$$\frac{\partial \mathcal{F}_s\{u\}}{\partial t} = -k\alpha^2\mathcal{F}_s\{u\} + k\alpha u|_{x=0}$$

We reduce the PDE to an ODE as

$$\frac{\partial \mathcal{F}_s\{u\}}{\partial t} + k\alpha^2\mathcal{F}_s\{u\} = u_0\alpha$$

Although nonhomogeneous, it is a first order linear ODE, and you can solve it by integral factor, which is $e^{\int k\alpha^2 dt} = e^{k\alpha^2 t}$ in this example, then the equation changes into

$$\frac{d}{dt}(e^{k\alpha^2 t}\mathcal{F}_s\{u\}) = u_0 k\alpha e^{k\alpha^2 t}$$

Integrate both sides to get

$$e^{k\alpha^2 t}\mathcal{F}_s\{u\} = \int u_0 k\alpha e^{k\alpha^2 t}dt = \frac{u_0 k\alpha}{k\alpha^2}e^{k\alpha^2 t} + C(\alpha) = \frac{u_0}{\alpha}e^{k\alpha^2 t} + C(\alpha)$$

$$\mathcal{F}_s\{u\} = \frac{u_0}{\alpha} + C(\alpha)(e^{-k\alpha^2 t})$$

To determine the $C(\alpha)$, we use the initial condition $u(x, 0) = 0$, so it is Fourier transform is also zero, therefore when we let $t = 0$, $\mathcal{F}_s\{u\}|_{t=0}$ should be zero. This forces $C(\alpha) = -\frac{u_0}{\alpha}$. So

$$\mathcal{F}_s\{u\} = \frac{u_0}{\alpha}[1 - e^{-k\alpha^2 t}]$$

Then get back to the inverse Fourier sine transform we get

$$u(x, t) = \frac{2}{\pi}\int_0^\infty \frac{u_0}{\alpha}[1 - e^{-k\alpha^2 t}]\sin\alpha x d\alpha = \frac{2}{\pi}\int_0^\infty \frac{u_0 \sin\alpha x}{\alpha}[1 - e^{-k\alpha^2 t}]d\alpha$$

□

15.4.6

Proof. It is a Neumann boundary condition, so use Fourier cosine transform to $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$, we get

$$\frac{\partial \mathcal{F}_c\{u\}}{\partial t} = -k\alpha^2\mathcal{F}_c\{u\} - k\frac{\partial u}{\partial x}|_{x=0}$$

We reduce to the ODE

$$\frac{\partial \mathcal{F}_c\{u\}}{\partial t} + k\alpha^2\mathcal{F}_c\{u\} = kA$$

Solve it by integral factor, which is $e^{\int k\alpha^2 dt} = e^{k\alpha^2 t}$ in this example, then the equation changes into

$$\frac{d}{dt}(e^{k\alpha^2 t}\mathcal{F}_c\{u\}) = kAe^{k\alpha^2 t}$$

Integrate both sides to get

$$e^{k\alpha^2 t}\mathcal{F}_c\{u\} = \int kAe^{k\alpha^2 t}dt = \frac{A}{\alpha^2}e^{k\alpha^2 t} + C(\alpha)$$

$$\mathcal{F}_c\{u\} = \frac{A}{\alpha^2} + C(\alpha)(e^{-k\alpha^2 t})$$

By the same reason as above $C(\alpha) = 0$ because of the initial condition. So

$$\mathcal{F}_c\{u\} = \frac{A}{\alpha^2}[1 - e^{-k\alpha^2 t}]$$

Then get back to the inverse Fourier cosine transform we get

$$u(x, t) = \frac{2}{\pi}\int_0^\infty \frac{A}{\alpha^2}[1 - e^{-k\alpha^2 t}]\cos\alpha x d\alpha = \frac{2}{\pi}\int_0^\infty \frac{A\cos\alpha x}{\alpha^2}[1 - e^{-k\alpha^2 t}]d\alpha$$

□

Fall 11, #6

Proof. The eigenvalues are n^2, and the solution has the form

$$u(x, t) = \sum_{n=1}^{\infty} C_n e^{-n^2 t}\sin nx$$

By initial condition $\sum_{n=1}^{\infty} C_n \sin nx = \sin 3x$, so $C_3 = 1$ and all others are zero. So $u(x,t) = e^{-9t}\sin 3x$. Then plug in $x = \frac{\pi}{6}$, $t = \frac{1}{2}$ you get $e^{-\frac{9}{2}}$. □

Fall 10, #7

Proof. Let $x = L$, $\sin \frac{n\pi}{L} x = \sin n\pi = 0$, so $u(L,t) = 0$. $u(0,t) = 0$ for similar reason. Therefore (b) is correct. □

Spring 11, #7

Proof. By separation of variables, $\frac{X'}{X} = -3\frac{Y'}{Y} = \lambda$, so $X = e^{\lambda x}$, $Y = e^{-3\lambda y}$. The solution has the form $Ce^{\lambda x - 3\lambda y}$. $u(0,1) = e$ implies $Ce^{-3\lambda} = e$, $u(1,0) = e^2$ implies $Ce^{\lambda} = e^2$, so by quotient $e^{-4\lambda} = e^{-1}$, $\lambda = \frac{1}{4}$, and plug back to get $C = e^{\frac{7}{4}}$. So $u(1,1) = e^{\frac{7}{4}} e^{\frac{1}{4} - 3\cdot\frac{1}{4}} = e^{\frac{5}{4}}$. □